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Ground-State and Quenched-State Properties of a One-Dimensional Interacting Lattice Gas in a Random Potential

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We determine the zero-temperature properties of a one-dimensional lattice gas of particles that interact via a nearest neighbor exclusion potential and are subject to a random external field. The model is a special limiting case of the random field Ising chain. We calculate (1) the energy and density of the ground state as well as the local energy-density correlation and (2) the pair correlation function. The latter calculation gives access to all higher order correlations. The structure factor is shown to be a squared Lorentzian. We also compare the ground state to the quenched state obtained by sequentially filling the lowest available energy levels.

KEY WORDS: Random field; Ising chain; correlation function.

1. INTRODUCTION

Magnetic lattice models in random external fields have been the subject of many investigations. Even for the simplest of these models, the random field Ising chain, results are usually limited to the free energy and other thermodynamic quantities. Here we study a special limiting case of a random field Ising chain for which, at zero temperature, the full set of correlation functions can be determined. The model is most naturally phrased in the language of a lattice gas.

We consider a one-dimensional lattice of N sites, $k = 1, 2, \ldots, N$. With each site k is associated a (negative) random potential $-\varepsilon_k$, as well as a binary variable σ_k, which is such that $\sigma_k = 0$ corresponds to the presence and $\sigma_k = 1$ to the absence of a particle at k. The particles interact via a

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nearest neighbor exclusion potential. Hence the Hamiltonian $\mathscr{H}$ of the system is

$$\mathscr{H} = -\sum_{k=1}^{N} \varepsilon_k \delta_{\sigma_k,0} \tag{1.1}$$

and configuration space consists of all $(\sigma_1, \sigma_2, ..., \sigma_N)$ satisfying $(1-\sigma_k)(1-\sigma_{k+1})=0$ for $k=1, 2, ..., N-1$. We shall take the ε_k to be independent random variables identically distributed according to

$$p(\varepsilon_k) = \begin{cases} 0, & \varepsilon_k < 0 \\ e^{-\varepsilon_k}, & \varepsilon_k \geqslant 0 \end{cases} \tag{1.2}$$

This model therefore has no free parameters. It can be obtained as a limiting case of the standard Ising chain in a random external field with Hamiltonian

$$\mathscr{H}_{\text{Ising}} = -J \sum_{k=1}^{N-1} s_k s_{k+1} - \sum_{k=1}^{N} H_k s_k \tag{1.3}$$

in which the s_k take the values ± 1. If one sets $s_k \equiv 1-2\sigma_k$ and takes the limit $J \to -\infty$ (extreme antiferromagnetism) while keeping the distributions of

$$\varepsilon_k \equiv 2H_k - 4J + 2J(\delta_{k,1} + \delta_{k,N}) \tag{1.4}$$

fixed, one recovers (apart from an infinite constant) the Hamiltonian (1.1) together with the condition of nearest neighbor exclusion.

The free energy of the Ising chain (1.3) has been studied in detail by Nieuwenhuizen and Luck[1] for a variety of distributions of the random field H_k. All distributions considered by these authors are combinations of exponential and delta functions, and the distribution (1.2) is again obtained in a special limit (not included in their treatment). Nieuwenhuizen and Luck present expressions for the free energy at all finite temperatures and obtain the zero-temperature energy and entropy in closed form. Their method (see also Nieuwenhuizen[2]) amounts to replacing the relevant integral equations by three-term recursion relations with nonrandom coefficients.

Even in one-dimensional random systems for which one can calculate the free energy, the evaluation of the pair correlation function is usually not feasible. An exception is provided by Grinstein and Mukamel,[3] who consider the Ising Hamiltonian (1.3) with a random field H_k having the particular distribution

$$H_k = \begin{cases} +\infty & \text{with probability } p/2 \\ 0 & \text{with probability } 1-p \\ -\infty & \text{with probability } p/2 \end{cases} \tag{1.5}$$

where p is a parameter in the interval [0, 1]. They find that the correlation function $\overline{\langle s_k s_l \rangle}$ (where $\langle \cdots \rangle$ is a thermal average and the overbar denotes averaging with respect to the randomness) behaves as

$$\overline{\langle s_k s_l \rangle} \simeq (A|k-l| + B)\, e^{-\kappa|k-l|}, \qquad |k-l| \to \infty \tag{1.6}$$

in which A, B, and κ are temperature-dependent constants. This behavior is distinctly different from the purely exponential decay laws that are the general rule for uniform one-dimensional systems with short-range interactions. It leads to a structure factor that is the sum of a Lorentzian and the square of a Lorentzian. Squared-Lorentzian type contributions to the correlation function are also found in mean-field theory (see Ref. 7).

In this work we limit ourselves to the distribution (1.2) and consider only the zero-temperature properties of the Hamiltonian (1.1). We extend, however, our calculations beyond the evaluation of the ground-state energy (Section 2) and of the particle density (Section 3.1). In Section 3.2 we derive the correlation between the energy and the occupation probability of a single site. In Section 4 we derive a fully explicit expression for the pair correlation function. In Section 5, finally, we introduce a kinetics for the system consisting in sequentially adding particles to the lowest available energy levels, until the nearest neighbor exclusion no longer allows the addition of any further particles. With the aid of results by Widom,[4] we calculate some of the properties of the quenched state thus obtained, and compare them to those of the ground state.

In Sections 2–4 our technique is the integral equation method common in the study of products of random matrices (its best known application in statistical mechanics probably is the study by McCoy and Wu[5] of the two-dimensional Ising model with uniaxially correlated randomness). The calculation of the ground-state energy leads to an integral recursion relation [Eq. (2.5)] for a sequence of functions $P_1(\alpha)$, $P_2(\alpha)$,... of one variable, whereas the calculation of the pair correlation function requires the solution of an integral recursion relation [Eq. (4.15)] for a sequence of functions $Q_1(\alpha, \beta, \gamma)$, $Q_2(\alpha, \beta, \gamma)$,... of three variables. We solve both recursions exactly. The determination of an arbitrary p-particle correlation function then merely amounts to the evaluation of a multiple integral of a product involving $p-1$ functions Q_n; the answer can again be given in closed form. All calculations of Sections 2–4 are for finite chains of N sites, which allows us to analyze the boundary and finite-size effects.

We summarize a few of our results. In the thermodynamic limit $N \to \infty$ the average ground-state particle density ρ^{G} and the average ground-state energy per site e^{G} take the values

$$e^{\mathrm{G}} = 4/9 \tag{1.7b}$$

$$e^{\mathrm{G}} = -2/3 \tag{1.7b}$$

These values may be compared to those for a densely packed chain, i.e., for a configuration $(\sigma_1,, \sigma_2, \sigma_3, \sigma_4,\ldots = (1, 0, 1, 0,\ldots)$, namely

$$\rho^{1010\ldots} = \tfrac{1}{2} \tag{1.8b}$$

$$e^{1010\ldots} = -\tfrac{1}{2}\int_0^\infty d\varepsilon\, \varepsilon e^{-\varepsilon} = -\tfrac{1}{2} \tag{1.8b}$$

Obviously the lower density of the ground state leads to a much better result for the energy. The results for the quenched state studied in Section 5, however, approach those for the ground state very closely. One

$$\rho^Q = \tfrac{1}{2}(1 - e^{-2}) = 0.4323\ldots \tag{1.9a}$$

$$e^Q = \int_0^1 dx\, e^{-2x} \log x = -0.6596\ldots \tag{1.9b}$$

The result (1.9a) is due to Widom[4] and is independent of the distribution $p(\varepsilon)$.

For the average particle–particle correlation function $g^G(m)$ between two sites a distance m apart and deep in the bulk we obtain

$$g^G(m) = \left(\frac{4}{9}\right)^2 + \frac{3m+5}{81}\left(-\frac{1}{2}\right)^{m-2}, \qquad m = 0, 1, 2,\ldots \tag{1.10}$$

This is of the form (1.6); moreover, the corresponding structure factor, given in Section 4, is a pure Lorentzian-squared! The surprising feature is that whereas Grinstein and Mukamel[3] find that the coefficient of the squared Lorentzian vanishes at zero temperature, in our model it is the only term present.

2. GROUND-STATE ENERGY

The calculation of the average free energy of a random lattice system is equivalent to the evaluation of a product of random transfer matrices. This problem can in turn be converted into an integral recursion relation. At zero temperature usually several of the ratios between the matrix elements of the transfer matrix vanish or become infinite, but in a slightly modified manner an integral recursion relation can nevertheless be derived. This will be shown below for the model defined by (1.1) and (1.2).

We consider for a given lattice site k the two quantities E_k^σ, $\sigma = 0, 1,$

defined as the minimum energy of the first k sites of the chain given that $\sigma_k = \sigma$. Explicitly,

$$E_k^\sigma = \min_{(\sigma_1,\ldots,\sigma_{k-1})} \left(-\sum_{j=1}^{k-1} \varepsilon_j \delta_{\sigma_j,0} - \varepsilon_k \delta_{\sigma,0} \right), \qquad k = 1,\ldots, N \tag{2.1}$$

where the minimum is on all $(\sigma_1,\ldots, \sigma_{k-1})$ such that $(\sigma_1,\ldots, \sigma_{k-1}, \sigma)$ satisfies the nearest neighbor exclusion principle. Since the E_k^σ depend on the random energy levels, they are also stochastic variables. By checking the various cases, one easily verifies that the E_k^σ satisfy the recusion relation

$$E_{k+1}^1 = \min(E_k^1, E_k^0) \tag{2.2a}$$

$$E_{k+1}^0 = -\varepsilon_{k+1} + E_k^1, \qquad k = 1, 2,\ldots, N-1 \tag{2.2b}$$

with initial condition $E_1^1 = 0$, $E_1^0 = -\varepsilon_1$. Upon subtracting these equations and defining

$$\xi_k = E_k^1 - E_k^0, \qquad k = 1, 2,\ldots, N \tag{2.3}$$

one finds that the ξ_k satisfy the one-variable recursion relation

$$\xi_{k+1} = \varepsilon_{k+1} + \min(0, -\xi_k), \qquad k = 1, 2,\ldots, N-1 \tag{2.4}$$

with initial condition $\xi_1 = \varepsilon_1$.

We shall denote by P_k the probability distribution of the variable ξ_k. The recursion relation (2.4) immediately allows us to write down an expression for P_{k+1} in terms of P_k, namely

$$P_{k+1}(\xi) = \int_{-\infty}^{\infty} d\xi'\, P_k(\xi') \times \int_{-\infty}^{\infty} d\varepsilon\, p(\varepsilon)\, \delta(\varepsilon + \min(0, -\xi') - \xi), \qquad k = 1, 2,\ldots, N-1 \tag{2.5}$$

The useful identity

$$\delta(x + \min(0, -y)) = \delta(x)\, \theta(-y) + \delta(x-y)\, \theta(y) \tag{2.6}$$

where

$$\theta(y) = \begin{cases} 0 & \text{for } y < 0 \\ 1 & \text{for } u \geqslant 0 \end{cases} \tag{2.7}$$

follows if one distinguishes the cases $y \geqslant 0$ and $y < 0$. Using (2.6), we find that Eq. (2.5) reduces to

$$P_{k+1}(\xi) = p(\xi) \int_{-\infty}^{0} d\xi' \, P_k(\xi') + \int_{0}^{\infty} d\xi' \, P_k(\xi') \, p(\xi + \xi'), \qquad k = 1, 2, \ldots, N-1 \tag{2.8}$$

The initial condition implied by our earlier discussion is

$$P_1(\xi) = p(\xi) \tag{2.9}$$

So far the function p could have been arbitrary. If one now uses the special form (1.2), one readily shows by induction that the solution is

$$P_k(\xi) = \begin{cases} \frac{1}{3}[1 - (-\frac{1}{2})^{k-1}] e^{\xi} & \text{for} \quad \xi < 0 \\ \frac{2}{3}[1 - (-\frac{1}{2})^{k}] e^{-\xi} & \text{for} \quad \xi \geqslant 0, \quad k = 1, 2, \ldots, N \end{cases} \tag{2.10}$$

For $k \to \infty$ the distributions (2.10) approach a limit P given by

$$P(\xi) = \begin{cases} \frac{1}{3} e^{\xi} & \text{for} \quad \xi < 0 \\ \frac{2}{3} e^{-\xi} & \text{for} \quad \xi \geqslant 0 \end{cases} \tag{2.11}$$

Having found this solution, we can now directly determine the average ground-state energy. It suffices to notice that upon using successive applications of the relation (2.2a) and the definition (2.3), as well as the fact that $E_1^1 = 0$, one can write the ground-state energy E_N^G of the total system as

$$\begin{aligned} E_N^G &\equiv \min(E_N^1, E_N^0) \\ &= E_N^1 + \min(0, -\xi_N) \\ &= \min(E_{N-1}^1, E_{N-1}^0) + \min(0, -\xi_N) \\ &= \sum_{j=1}^{N} \min(0, -\xi_j) \end{aligned} \tag{2.12}$$

One can therefore calculate the average ground-state energy per site e_N^G as

$$e_N^G \equiv N^{-1} \sum_{k=1}^{N} \int_{-\infty}^{\infty} d\xi \, P_k(\xi) \min(0, -\xi) \tag{2.13}$$

Since in the limit $k \to \infty$ the distributions P_k tend to a limit P, the average ground-state energy per site in the thermodynamic limit becomes

$$e^{\mathrm{G}} = \lim_{N \to \infty} e_N^{\mathrm{G}} = \int_{-\infty}^{\infty} d\xi\, P(\xi) \min(0, -\xi) \tag{2.14}$$

Inserting now the explicit functions (2.10) and (2.11) in (2.13) and (2.14), one finds that

$$e_N^{\mathrm{G}} = -\tfrac{2}{3} - \tfrac{2}{9} N^{-1}[1 - (-\tfrac{1}{2})^N], \qquad N = 1, 2, \ldots \tag{2.15a}$$

$$e^{\mathrm{G}} = -\tfrac{2}{3} \tag{2.15b}$$

the latter being the result announced in the introduction. The equations (2.15) are consistent with the intuitive notion that systems with odd N should have a slightly lower average ground-state energy per site than systems with even N, and that for finite N the average ground-state energy per site is slightly lower than the limit value $e^{\mathrm{G}} = -2/3$.

3. PARTICLE DENSITY AND ON-SITE ENERGY-DENSITY CORRELATION

3.1. Particle Density

We shall denote the ground-state configuration as $(\sigma_1^{\mathrm{G}}, \sigma_2^{\mathrm{G}}, \ldots, \sigma_N^{\mathrm{G}})$. The σ_k^{G} are functions of the set of random levels $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$. Obviously the ground state must be built up of two types of sequences, 01 and 011: the particle interaction interaction forbids the occurrence of two successuve 0's (occupied sites), and if three consecutive 1's (empty sites) occurred, the energy could always be lowered by changing the middle one into a 0.

The local particle density $\rho_N^{\mathrm{G}}(k)$ at lattice site k is the analog of the local magnetization for the Ising chain. It is defined as

$$\rho_N^{\mathrm{G}}(k) = 1 - \overline{\sigma_k^{\mathrm{G}}}, \qquad k = 1, \ldots, N \tag{3.1}$$

where the overbar indicates an average over all $(\varepsilon_1, \ldots, \varepsilon_N)$. In this section we shall calculate $\rho_N^{\mathrm{G}}(k)$. We begin by defining E^σ (for $\sigma = 0, 1$) as the minimum energy of the system subject to the condition that $\sigma_k = \sigma$. Clearly, $\overline{\sigma_k^{\mathrm{G}}}$ is equal to the fraction of all energy level sets $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ for which E^1 is less than E^0. Explicitly,

$$\overline{\sigma_k^{\mathrm{G}}} = \int_{-\infty}^{\infty} \prod_{j=1}^{N} d\varepsilon_j\, p(\varepsilon_j)\, \theta(E^0 - E^1), \qquad k = 1, \ldots, N \tag{3.2}$$

The answer for the spcial average $\overline{\sigma_N^G}$ follows immediately from the results of the previous section, since it is equal to the probability of $\xi_N = E_N^1 - E_N^0$ being negative. From the probability distribution $P_N(\xi_N)$ given in (2.10) we therefore have that

$$\overline{\sigma_N^G} = \tfrac{1}{3}[1 - (-\tfrac{1}{2})^{N-1}]$$

Using (3.1) and the fact that by symmetry $\rho_N^G(1) = \rho_N^G(N)$, we have

$$\rho_N^G(1) = \rho_N^G(N) = \tfrac{2}{3}[1 - (-\tfrac{1}{2})^N] \tag{3.3}$$

In the general case $k = 2,..., N-1$ the calculation is more complicated. Let E_{k-1}^σ be the minimum energy of the variables σ_1 through σ_{k-1} subject to $\sigma_{k-1} = \sigma$ (which is just the definition of Section 2) and let $\bar{E}_{N-k}^\sigma$ analogously be the minimum energy of the variables σ_{k+1} through σ_N subject to $\sigma_k = \sigma$. We can then decompose E^0 and E^1 as

$$E^0 = E_{k-1}^1 - \varepsilon_k + \bar{E}_{N-k}^1 \tag{3.4a}$$

$$E^1 = \min(E_{k-1}^0, E_{k-1}^1) + \min(\bar{E}_{N-k}^0, \bar{E}_{N-k}^1), \qquad k = 2,..., N-1 \tag{3.4b}$$

When subtracting these equations, we find

$$E^1 - E^0 = \varepsilon_k + \min(0, -\xi) + \min(0, -\eta), \qquad k = 2,..., N-1 \tag{3.5}$$

where we have defined the differences

$$\xi = E_{k-1}^1 - E_{k-1}^0 \tag{3.6a}$$

$$\eta = \bar{E}_{N-k}^1 - \bar{E}_{N-k}^0, \qquad k = 2,..., N-1 \tag{3.6b}$$

The quantities ξ and η are independent random variables whose distributions $P_{k-1}(\xi)$ and $P_{N-k}(\eta)$, respectively, have been calculated in Section 2. Upon using these results in the expression (3.2) for $\overline{\sigma_k^G}$, we find

$$\overline{\sigma_k^G} = \int_{-\infty}^{\infty} d\xi\, P_{k-1}(\xi) \int_{-\infty}^{\infty} d\eta\, P_{N-k}(\eta)$$

$$\times \int_{-\infty}^{\infty} d\varepsilon\, p(\varepsilon)\, \theta(-\varepsilon - \min(0, -\xi) - \min(0, -\eta\eta)), \qquad k = 2,..., N-1 \tag{3.7}$$

After working out the theta function in a way analogous to (2.6), using the explicit distribution (1.2) for $p(\varepsilon)$, carrying out the integration over ε, and using (3.1), we obtain the intermediate result

$$\rho_N^G(k) = \int_{-\infty}^{\infty} d\xi\, P_{k-1}(\xi) \int_{-\infty}^{\infty} d\eta\, P_{N-k}(\eta)\, \Theta(\xi)\, \Theta(\eta), \qquad k = 2,..., N-1 \tag{3.8}$$

where we introduced the abbreviation

$$\Theta(x) = \theta(-x) + \theta(x)e^{-x} \tag{3.9}$$

After substitution of (3.9) into (3.8), one easily performs the ξ and η integrations and finds a final results for $\rho_N^G(k)$, which also covers the special cases (3.3), namely

$$\rho_N^G(k) = \tfrac{4}{9}[1 - (-\tfrac{1}{2})^k][1 - (-\tfrac{1}{2})^{N-k+1}], \qquad k = 1, 2, ..., N, \quad N = 1, 2, ... \tag{3.10}$$

Hence the bulk density is

$$\rho^G \equiv \lim_{\substack{k \to \infty \\ N-k \to \infty}} \rho_N^G(k) = 4/9 = 0.444... \tag{3.11}$$

which is the result (1.7a). It shows that the ground-state configuration is different from the maximum density configuration, which has $\rho = \frac{1}{2}$.

Another result of interest that can be derived from (3.10) is the density $\rho_{\text{bound}}^G(k)$ near one of the system boundaries,

$$\rho_{\text{bound}}^G(k) \equiv \lim_{N \to \infty} \rho_N^G(k) = \tfrac{4}{9}[1 - (-\tfrac{1}{2})^k] \tag{3.12}$$

The number of excess particles n_{exc} near one boundary therefore is

$$n_{\text{exc}} \equiv \sum_{k=1}^{\infty} [\rho_{\text{bound}}^G(k) - \rho^G] = 4/27 \tag{3.13}$$

Furthermore, the total density of the finite system is

$$\rho_N^G \equiv N^{-1} \sum_{k=1}^{N} \rho_N^G(k) = \tfrac{4}{9} + \tfrac{8}{27} N^{-1} + O(2^{-N}) \tag{3.14}$$

The finite-size correction is, up to terms that vanish exponentially for $N \to \infty$, equal to $2n_{\text{exc}}/N$, and therefore is entirely due to the two boundaries.

3.2. On-Site Energy-Density Correlation

Suppose the potential $-\varepsilon_k$ at the site k has the value $-\varepsilon$. We wish to find the probability $f_N^G(k; \varepsilon)$ that this site is occupied by a particle. The expression to be calculated is

$$f_N^G(k; \varepsilon) = \overline{\delta(\varepsilon_k - \varepsilon)(1 - \sigma_k^G)} / \overline{\delta(\varepsilon_k - \varepsilon)}, \qquad k = 1, ..., N \tag{3.15}$$

in which the numerator is the probability that both $\varepsilon_k = \varepsilon$ and site k is occupied. The evaluation of (3.15) proceeds along the lines of the previous subsection. After the integration over ε_k one obtains the intermediate result

$$\overline{\delta(\varepsilon_k - \varepsilon)\sigma_k^{\mathrm{G}}} = e^{-\varepsilon} \int_{-\infty}^{\infty} d\xi\, P_{k-1}(\xi) \int_{-\infty}^{\infty} d\eta\, P_{N-k}(\eta)$$
$$\times [\theta(-\varepsilon+\xi)\,\theta(\xi)\,\theta(-\eta) + \theta(-\varepsilon+\eta)\,\theta(-\xi)\,\theta(\eta)$$
$$+ \theta(-\varepsilon+\xi+\eta)\,\theta(\xi)\,\theta(\eta)], \qquad k = 2,\ldots, N-1 \tag{3.16}$$

Upon using the expressions (2.10) for P_{k-1} and P_{N-k}, one can perform the integrations over ξ and η. The cases $k=1$ and $k=N$ again require separate consideration.

We give the final result only for a site k deep in the bulk:

$$f^{\mathrm{G}}(\varepsilon) \equiv \lim_{\substack{k\to\infty \\ N-k\to\infty}} f_N^{\mathrm{G}}(k;\varepsilon) = 1 - \tfrac{4}{9}(2+\varepsilon)e^{-\varepsilon}, \qquad \varepsilon \geqslant 0 \tag{3.17}$$

For $\varepsilon \to \infty$ the quantity $f^{\mathrm{G}}(\varepsilon)$ approaches unity, as expected: sites with very low potentials are occupied with a probability close to one. Finally, one can verify with the aid of (3.17) that

$$\int_0^{\infty} d\varepsilon\, e^{-\varepsilon} f^{\mathrm{G}}(\varepsilon) = \rho^{\mathrm{G}} \tag{3.18}$$

as it should. In Section 5 we compare the function $f^{\mathrm{G}}(\varepsilon)$ to its quenched-state analog $f^{\mathrm{Q}}(\varepsilon)$.

4. GROUND-STATE PAIR CORRELATION FUNCTION

We wish to calculate the average ground-state particle–particle correlation function

$$g_N^{\mathrm{G}}(k, l) \equiv \overline{(1-\sigma_k^{\mathrm{G}})(1-\sigma_l^{\mathrm{G}})}, \qquad 1 \leqslant k < l \leqslant N \tag{4.1}$$

For the special case $l = k+1$ the nearest neighbor exclusion condition $(1-\sigma_k^{\mathrm{G}})(1-\sigma_{k+1}^{\mathrm{G}}) = 0$ directly gives

$$g_N^{\mathrm{G}}(k, k+1) = 0, \qquad k = 1, 2,\ldots, N-1 \tag{4.2}$$

The general case, however, requires more effort. In Section 4.1 we derive the integral recursion relation that constitutes the hard core of this problem. It is a nontrivial generalization of the recursion considered in Section 2. In Section 4.2 we solve this new recursion relation and calculate the particle–particle correlation.

4.1. Reduction to an Integral Recursion Relation

We define the quantity $E^{\sigma\tau}$ (where σ and τ take the values 0 or 1) as the minimum energy of the system subject to the condition that

$$\sigma_k = \sigma, \qquad \sigma_l = \tau \tag{4.3}$$

Clearly, $\overline{\sigma_k^G \sigma_l^G}$ is equal to the fraction of energy level sets $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ for which E^{11} is less than the other three $E^{\sigma\tau}$. Explicitly,

$$\overline{\sigma_k^G \sigma_l^G} = \int_{-\infty}^{\infty} \prod_{j=1}^{N} d\varepsilon_j \, p(\varepsilon_j) \, \theta(E^{10} - E^{11}) \times \theta(E^{01} - E^{11}) \, \theta(E^{00} - E^{11}), \qquad 1 \leqslant k < l \leqslant N \tag{4.4}$$

Hence, in order to find $\overline{\sigma_k^G \sigma_l^G}$, it is sufficient to derive from the distribution of the ε_j the joint probability distribution of the arguments of three three theta functions.

It is useful to define $E_{l-k-1}^{\sigma\tau}$, for $k < l-1$, as the minimum energy of the sites $k+1$ through $l-1$ under the condition (4.3). Using again the definitions of E_k^σ and $\bar{E}_{N-k}^\sigma$ given in Section 3, we can decompose $E^{\sigma\tau}$ as

$$E^{\sigma\tau} = E_k^\sigma + E_{l-k-1}^{\sigma\tau} + \bar{E}_{N-l+1}^\tau, \qquad 1 \leqslant k < l-1 \leqslant N-1 \tag{4.5}$$

Instead of the four variables $E^{\sigma\tau}$, we shall now consider only the three differences that occur in Eq. (4.4). In order to simplify the notation, we define

$$\begin{aligned} \xi &= E_k^1 - E_k^0 \\ \eta &= \bar{E}_{N-l+1}^1 - \bar{E}_{N-l+1}^0, \qquad 1 \leqslant k < l-1 \leqslant N-1 \end{aligned} \tag{4.6}$$

$$\begin{aligned} \alpha_n &= E_n^{11} - E_n^{10} \\ \beta_n &= E_n^{01} - E_n^{00} \\ \gamma_n &= E_n^{11} - E_n^{01}, \qquad n = l-k-1, \quad 1 \leqslant k < l-1 \leqslant N-1 \end{aligned} \tag{4.7}$$

For the energy differences in Eq. (4.4) we then find, with the aid of Eqs. (4.5)–(4.7),

$$\begin{aligned} E^{10} - E^{11} &= -\alpha_{l-k-1} - \eta \\ E^{01} - E^{11} &= -\xi - \gamma_{l-k-1} \\ E^{01} - E^{11} &= -\xi - \gamma_{l-k-1} \\ E^{00} - E^{11} &= -\xi - \beta_{l-k-1} - \gamma_{l-k-1} - \eta, \qquad 1 \leqslant k < l-1 \leqslant N-1 \end{aligned} \tag{4.8}$$

The quantities ξ and η are analogous to the variables ξ_j discussed in Section 2; they therefore have the probability distributions $P_k(\xi)$ and $P_{N-l+1}(\eta)$, respectively. The quantities α_n, β_n, and γ_n are determined by ε_{k+1} through ε_{k+n} and therefore have a joint probability distribution that we shall call $Q_n(\alpha_n, \beta_n, \gamma_n)$, and which is still unknown. Upon introducing all this into expression (4.4), we obtain

$$\begin{aligned}\overline{\sigma_k^{\mathrm{G}}\sigma_l^{\mathrm{G}}} = &\int_{-\infty}^{\infty} d\xi\, P_k(\xi) \int_{-\infty}^{\infty} d\eta\, P_{N-l+1}(\eta) \\ &\times \int_{-\infty}^{\infty} d\alpha\, d\beta\, d\gamma\, Q_{l-k-1}(\alpha, \beta, \gamma) \\ &\times \theta(-\xi-\alpha)\,\theta(-\eta-\gamma)\,\theta(-\gamma-\eta-\beta-\gamma), \qquad 1 \leqslant k < l-1 \leqslant N-1\end{aligned} \tag{4.9}$$

It is clear that for the higher correlation functions similar relations can be written down. For example, $\overline{\sigma_k^{\mathrm{G}}\sigma_l^{\mathrm{G}}\sigma_m^{\mathrm{G}}}$ (with $1 \leqslant k < l-1 < m-2 \leqslant N-2$) would be represented by an integral involving the functions Q_{l-k-1} and Q_{m-l-1}. Since the functions P_j are known [Eq. (2.10)], we could at this stage carry out the integrals on ξ and η in expression (4.9). It will be more convenient, however, first to determine the functions Q_n and then to do the integrals on α, β, and γ.

The variables $E_n^{\sigma\tau}$ depend on $\varepsilon_{k+1}, \varepsilon_{k+2}, \ldots, \varepsilon_{k+n}$. One can relate the $E_n^{\sigma\tau}$ to the $E_{n-1}^{\sigma\tau'}$ (at fixed σ and for $\tau, \tau' = 0, 1$) in the same way as in Section 2 the E_k^{τ} were related to the $E_{k-1}^{\tau'}$. (Note, however, that the energy of the site variable constrained to have the value τ at the end of the sequence is included in the definition of E_k^{τ}, whereas it is not in the definition of $E_n^{\sigma\tau}$.) Upon considering the different cases, one finds

$$E_n^{\sigma 1} = \min(-\varepsilon_{k+n} + E_{n-1}^{\sigma 0}, E_{n-1}^{\sigma 1}) \tag{4.10a}$$

$$E_n^{\sigma 0} = E_{n-1}^{\sigma 1}, \qquad \sigma = 0, 1; \quad n = 2, 3, \ldots, N-2 \tag{4.10b}$$

The initial condition is

$$E_1^{11} = -\varepsilon_{k+1}, \qquad E_1^{\sigma\tau} = 0 \quad \text{for} \quad \sigma\tau = 10, 01, 00 \tag{4.11}$$

By forming the appropriate differences of these relations and using the definitions (4.7), one obtains a recursion relation expressing $\alpha_n, \beta_n, \gamma_n$ in $\alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}$:

$$\begin{aligned}\alpha_n &= \min(0, -\varepsilon_{k+n} - \alpha_{n-1}) \\ \beta_n &= \min(0, -\varepsilon_{k+n} - \beta_{n-1}) \\ \gamma_n &= \gamma_{n-1} + \min(0, -\varepsilon_{k+n} - \alpha_{n-1}) - \min(0, -\varepsilon_{k+n} - \beta_{n-1})\end{aligned} \tag{4.12}$$

The initial condition is

$$\alpha_1 = -\varepsilon_{k+1}, \qquad \beta_1 = 0, \qquad \gamma_1 = -\varepsilon_{k+1} \tag{4.13}$$

This relation allows us to express Q_n in Q_{n-1} according to

$$\begin{aligned} Q_n(\alpha, \beta, \gamma) = & \int_{-\infty}^{\infty} d\varepsilon_{k+n}\, p(\varepsilon_{k+n}) \\ & \times \int_{-\infty}^{\infty} d\alpha_{n-1}\, d\beta_{n-1}\, d\gamma_{n-1}\, Q_{n-1}(\alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}) \\ & \times \delta(\alpha_n - \alpha)\, \delta(\beta_n - \beta)\, \delta(\gamma_n - \gamma) \end{aligned} \tag{4.14}$$

in which α_n, β_n, and γ_n are given by (4.12). If we substitute (4.12) into (4.14), perform the integration on γ_{n-1}, and use (1.2) and (2.6), we can cast the integral recurrence in the form

$$\begin{aligned} Q_{n+1}(\alpha, \beta, \gamma) \equiv & \mathscr{R} Q_n(\alpha, \beta, \gamma) \\ = & \theta(-\alpha)\, \theta(-\beta) \int_0^{\infty} d\varepsilon\, e^{-\varepsilon} Q_n(-\alpha-\varepsilon, -\beta-\varepsilon, \gamma+\beta-\alpha) \\ & + \theta(-\alpha)\, \delta(\beta) \int_0^{\infty} d\varepsilon\, e^{-\varepsilon} \int_{-\infty}^{-\varepsilon} d\beta'\, Q_n(-\alpha-\varepsilon, \beta', \gamma-\alpha) \\ & + \delta(\alpha)\, \theta(-\beta) \int_0^{\infty} d\varepsilon\, e^{-\varepsilon} \int_{-\infty}^{-\varepsilon} d\alpha'\, Q_n(\alpha', -\beta-\varepsilon, \gamma+\beta) \\ & + \delta(\alpha)\, \delta(\beta) \int_0^{\infty} d\varepsilon\, e^{-\varepsilon} \int_{-\infty}^{-\varepsilon} d\alpha' \int_{-\infty}^{-\varepsilon} d\beta'\, Q_n(\alpha', \beta', \gamma) \\ & (n = 1, 2, \ldots, N-3) \end{aligned} \tag{4.15}$$

From (4.13) and the explicit form (1.2) of $p(\varepsilon)$, we find the initial condition

$$Q_1(\alpha, \beta, \gamma) = \delta(\alpha - \gamma)\, \delta(\beta)\, \theta(-\alpha) e^{\alpha} \tag{4.16}$$

This completes the derivation of the recursion relation for Q_n. By integrating both sides of Eq. (4.15) on γ, one obtains an integral recursion relation for the functions $\int_{-\infty}^{\infty} d\gamma\, Q_n(\alpha, \beta, \gamma)$; by integrating on both γ and β, one finds a recursion relation closely related to (2.5).

4.2. Solution of the Recursion Relation for Q_n

It is useful first to point out a symmetry property that the solutions $Q_2, Q_3, \ldots$ to the recursion relation (4.15) should possess. We define

$$\zeta_n \equiv \beta_n + \gamma_n = E_n^{11} - E_n^{00} \tag{4.17}$$

and write Q_n as a function of α_n, γ_n, and ζ_n:

$$Q_n(\alpha_n, \gamma_n; \zeta_n) \equiv Q_n(\alpha_n, \beta_n, \gamma_n) \tag{4.18}$$

Since the ε's are all identically distributed, the function Q_n should be invariant if in (4.3) we interchange σ and τ. In view of the definitions (4.7) and (4.17), this corresponds to an interchange of α_n and γ_n, while ζ_n transforms into itself. Hence, we should have

$$Q_n(\alpha, \gamma; \zeta) = Q_n(\gamma, \alpha; \zeta) \tag{4.19}$$

This relation is indeed satisfied by the initial condition, which in the new notation reads

$$Q_1(\alpha, \gamma; \zeta) = \delta(\alpha - \gamma)\, \delta(\gamma - \zeta)\, \theta(-\alpha) e^{\alpha} \tag{4.20}$$

The symmetry (4.19) is not obviously conserved by the recyrsion relation (4.15), and therefore its conservation will serve as a check on the further calculations.

It is tedious but straightforward to use (4.15) to calculate the first few iterates of Q_1. One then observes, and can subsequently check by induction, that each of them can be written as a linear combination of the following basis functions:

$$F_1(\alpha, \gamma; \zeta) = \delta(\alpha)\, \delta(\gamma)\, \delta(\zeta) \tag{4.21a}$$

$$F_2(\alpha, \gamma; \zeta) = \tfrac{1}{2}\{\delta(\gamma)\, \delta(\zeta - \alpha)\, \theta(-\alpha) e^{\alpha} + [\alpha \leftrightarrow \gamma]\} \tag{4.21b}$$

$$F_3(\alpha, \gamma; \gamma) = \theta(-\alpha)\, \theta(-\gamma)\, \delta(\zeta - \alpha - \gamma) e^{\alpha + \gamma} \tag{4.21c}$$

$$\begin{aligned} G_n(\alpha, \gamma;\ \zeta) = {} & \{\delta(\alpha)\, \theta(-\gamma)\, \theta(\gamma - \zeta)\, e^{n\zeta - (n-1)\gamma} + [\alpha \leftrightarrow \gamma]\} \\ & + (n-2)\, \theta(-\alpha)\, \theta(-\gamma)\, \theta(\alpha + \gamma - \zeta) \\ & \times e^{n\zeta - (n-1)(\alpha + \gamma)}, \qquad n = 2, 4, 6, \ldots \end{aligned} \tag{4.21d}$$

$$\begin{aligned} H_n(\alpha, \gamma; \zeta) = {} & n^{-1}\, \delta(\alpha - \gamma)\, \delta(\zeta - \alpha)\, \theta(-\alpha) e^{n\alpha} \\ & + n^{-1}(n-1)\{\delta(\zeta - \alpha)\, \theta(\gamma - \alpha)\, \theta(-\gamma)\, e^{\alpha + (n-1)\gamma} + [\alpha \leftrightarrow \gamma]\} \\ & + n^{-1}(n-1)^2\, \theta(\alpha - \zeta)\, \theta(\gamma - \zeta)\, \theta(\zeta - \alpha - \gamma) \\ & \times e^{-(n-2)\zeta + (n-1)(\alpha + \gamma)}, \qquad n = 1, 3, 5, \ldots \end{aligned} \tag{4.21e}$$

Each of these functions has been normalized such that it yields 1 upon integration over α, γ, and ζ. Furthermore, they are all symmetric in α and γ. The first few Q_n thus calculated are

$$
\begin{aligned}
Q_1 &= H_1 \\
Q_2 &= G_2 \\
Q_3 &= \tfrac{3}{4}H_3 + \tfrac{1}{4}F_1 \\
Q_4 &= \tfrac{1}{2}G_4 + \tfrac{1}{2}F_2 \\
Q_5 &= \tfrac{5}{16}H_5 + \tfrac{3}{16}F_1 + \tfrac{1}{4}F_2 + \tfrac{1}{4}F_3
\end{aligned}
\tag{4.22}
$$

In order to find the general Q_n it is convenient to calculate the action of $\mathscr{R}$ on the basis functions. The calculation is again a straightforward application of the definition (4.15) of $\mathscr{R}$. It is now necessary to distinguish F_2^+ and F_2^-, defined as the first and the second terms, respectively, in $\{\cdots\}$ in (4.21b). Hence

$$
F_2 = \tfrac{1}{2}(F_2^+ + F_2^-) \tag{4.23}
$$

The result for the action of $\mathscr{R}$ then is

$$
\begin{aligned}
&\mathscr{R}F_1 = F_2^+ \\
&\mathscr{R}F_2^+ = \tfrac{1}{2}(F_1 + F_2^+) \\
&\mathscr{R}F_2^- = F_3 \\
&\mathscr{R}F_3 = \tfrac{1}{2}(F_2^- + F_3) \\
&\mathscr{R}G_n = \tfrac{1}{2}n^{-1}(n+1)H_{n+1} + \tfrac{1}{2}n^{-1}F_1 + \tfrac{1}{2}n^{-1}(n-2))F_2^-, \qquad n = 2, 4, 6,\ldots \\
&\mathscr{R}H_n = \tfrac{1}{2}n^{-1}(n+1)G_{n+1} + \tfrac{1}{2}n^{-1}(n-1)F_2^-, \qquad n = 1, 3, 5,\ldots
\end{aligned}
\tag{4.24}
$$

Upon taking $Q_1 = H_1$ as the initial condition, one finds the general solution

$$
\begin{aligned}
Qn &= 2^{-n+1}nH_n + \tfrac{1}{9}[1 + (3n-4)2^{-n+1}]F_1 \\
&\quad + \tfrac{4}{9}[1 - (3n-1)2^{-n}](F_2 + F_3), \qquad n = 1, 3, 5,\ldots
\end{aligned}
\tag{4.25a}
$$

$$
\begin{aligned}
Q_n &= 2^{-n+1}nG_n + \tfrac{1}{9}[1 - (3n-4)2^{-n+1}](F_1 + 4F_3) \\
&\quad + \tfrac{4}{9}[1 + (3n-10)2^{-n}]F_2, \qquad n = 2, 4, 6,\ldots
\end{aligned}
\tag{4.25b}
$$

One easily checks that for each Q_n the coefficients of the basis functions on the right-hand side of Eqs. (4.25) sum to unity, as they should.

These results are now to be substituted into (4.9), after which the correlation $\overline{\sigma_k^G \sigma_l^G}$ can be computed directly. It is useful to have the results at one intermdiate stage. We therefore definite for any function $A(\alpha, \gamma; \zeta)$

$$
\begin{aligned}
\tilde{A}(\xi, \eta) &= \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\gamma \int_{-\infty}^{\infty} d\zeta \\
&\quad \times A(\alpha, \gamma; \zeta)\, \theta(-\xi - \alpha)\, \theta(-\eta - \gamma)\, \theta(-\xi - \eta - \zeta)
\end{aligned}
\tag{4.26}
$$

Also using the definition (3.9), we find by straightforward integration

$$\begin{aligned}
\tilde{F}_1(\xi,\eta) &= \theta(-\xi)\,\theta(-\eta) \\
\tilde{F}_2(\xi,\eta) &= \tfrac{1}{2}\{\theta(-\xi)\,\Theta(\eta)+\theta(-\eta)\,\Theta(\xi)\} \\
\tilde{F}_3(\xi,\eta) &= \Theta(\xi)\,\Theta(\eta) \\
\tilde{G}_n(\xi,\eta) &= n^{-1}[\theta(-\xi)\,\Theta(\eta)+\theta(-\eta)\,\Theta(\xi)]+n^{-1}(n-2)\,\Theta(\xi)\,\Theta(\eta) \\
\tilde{H}_n(\xi,\eta) &= \Theta(\xi)\,\Theta(\eta)
\end{aligned} \tag{4.27}$$

If the results (4.25)–(4.27) are combined and substituted into (4.9), the averages with respect to $P_k(\xi)$ and $P_{N-l+1}(\eta)$ are easily evaluated with the aid of

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\xi\, P_s(\xi)\,\theta(-\xi) = \tfrac{1}{3}[1-(-\tfrac{1}{2})^{s-1}] \\
&\int_{-\infty}^{\infty} d\xi\, P_s(\xi)\,\Theta(\xi) = \tfrac{2}{3}[1-(-\tfrac{1}{2})^{s+1}], \qquad s=1,2,\ldots,N
\end{aligned} \tag{4.28}$$

The final result also holds in the special cases $k=l-1$ and $k=l$, and reads

$$\begin{aligned}
\overline{\sigma_k^{\mathrm{G}}\sigma_l^{\mathrm{G}}} = {} & (1/81)\{25+[3(l-k)+5](-\tfrac{1}{2})^{l-k-2}\} \\
& -(2/81)\{20-[3(l-k)-4](-\tfrac{1}{2})^{l-k-2}\} \\
& \times[(-\tfrac{1}{2})^{k+1}+(-\tfrac{1}{2})^{N-l+2}] \\
& +(4/81)\{16+[3(l-k)-13](-\tfrac{1}{2})^{l-k-2}\} \\
& \times(-\tfrac{1}{2})^{N-l+k+3}, \qquad 1\leqslant k\leqslant l\leqslant N
\end{aligned} \tag{4.29}$$

The special case of greatest interest is the expression for the particle–particle correlation function in the bulk. From (4.1) and (4.29) together with the expressions (3.1) and (3.10) we obtain

$$\begin{aligned}
g^{\mathrm{G}}(m) &\equiv \lim_{\substack{k\to\infty \\ N-k-m\to\infty}} g_N^{\mathrm{G}}(k,k+m) \\
&= (1/81)[16+(3m+5)(-\tfrac{1}{2})^{m-2}], \qquad m=0,1,2,\ldots
\end{aligned} \tag{4.30}$$

which gives (1.10). Alternatively, by taking the limit $N\to\infty$ at finite k and l one easily specializes (4.29) to the correlation between a site k on or near the boundary and an other, arbitrary site l.

We finally discuss the structure factor in the bulk, which is defined by

$$S^{\mathrm{G}}(q) \equiv \sum_{m=-\infty}^{\infty} e^{iqm}[g^{\mathrm{G}}(|m|)-g^{\mathrm{G}}(\infty)] \tag{4.31}$$

Because of the nearest neighbor exclusion, it is to be expected that $S^G(q)$ will have a maximum near $q=\pi$. We therefore consider $S^G(\pi+q)$, for which a direct calculation from (4.30) and (4.31) gives

$$S^G(\pi+q)=\frac{4}{3}\frac{1}{[1+8\sin^2(\frac{1}{2}q)]^2} \tag{4.32}$$

Remarkably, this is (at least in the small-q limit) a *pure Lorentzian-squared.* We have thus confirmed once more the presence of such terms in the structure factor of random systems.

5. PROPERTIES OF THE QUENCHED STATE

In this section we go beyond the study of equilibrium properties. We imagine that an initially empty lattice is sequentially filled; each new particle is placed on the site with lowest energy still available subject to the condition of nearest neighbor exclusion. We shall refer to the jammed state obtained when no further particles can be added to the lattice as the "quenched state." Obviously, other prescriptions for arriving at quenched states are thinkable; the one given here corresponds most closely to an infinitely rapid quench from a high-temperature state of near zero density to a zero-temperature state of high density.

One can ask the same questions about the quenched state that we studied for the ground state. We limit ourselves to the quenched-state density ρ^Q, the quenched-state energy e^Q, and the fraction $f^Q(\varepsilon)$ of sites with energy $-\varepsilon$ that in the quenched state are occupied.

At the basis of our results for the quenched state is the work by Widom,[4] who considers[4,6] a variety of random sequential filling problems. Among these is precisely the one of sequentially placing particles on a one-dimensional lattice (but in the absence of a site potential $-\varepsilon_k$) such that nearest neighbor occupancy cannot occur. Widom[4] shows that in the final quenched configuration on an infinite linear lattice a fraction

$$\rho^Q=\tfrac{1}{2}(1-e^{-2}) \tag{5.1}$$

of all sites is occupied.

For the system of interest to us, the order in which the sites are filled is determined by the set of preassigned energies $(\varepsilon_1,\ldots,\varepsilon_N)$. But at each stage of the filling process any one of the available sites (i.e., sites not yet occupied themselves and not having either neighbor occupied) may have the lowest energy with equal probability. Therefore, the filling is a random process identical to Widom's, and the result (5.1), reproduced in Eq. (1.9a), holds independent of the energy distribution $p(\varepsilon)$.

In the limit of an infinite lattice the energy per site e^{Q} of the quenched state can be expressed as

$$e^{Q} = -\int_{-\infty}^{\infty} d\varepsilon\, \varepsilon p(\varepsilon)\, f^{Q}(\varepsilon) \tag{5.2}$$

in which the distribution $p(\varepsilon)$ is still arbitrary. In order to determine $f^{Q}(\varepsilon)$ and e^{Q}, we suppose that the filling process has proceeded to a point where a fraction ρ of all sites has been occupied. There exists, then, an as yet unknown value $\varepsilon(\rho)$ such that no sites with energies less than $-\varepsilon(\rho)$ are available. Furthermore, there exists a $\phi(\rho)$ such that at this point the number of available sites is a fraction $\phi(\rho)$ of all lattice sites. The function $\phi(\rho)$ plays an important role in Widom's calculation,[4] were it is shown to be given by

$$\phi(\rho) = (1-2\rho)[1 + \tfrac{1}{2}\log(1-2\rho)] \tag{5.3}$$

Since all available sites have energies $-\varepsilon_k$ larger than $-\varepsilon(\rho)$, and since for these there is no correlation between their energy and their availability. we have that

$$f^{Q}(\varepsilon) = \phi(\rho(\varepsilon))/P(\varepsilon) \tag{5.4}$$

where $\rho(\varepsilon)$ is the inverse function of $\varepsilon(\rho)$ and we introduce the abbreviation

$$P(\varepsilon) = \int_{-\infty}^{\varepsilon} d\varepsilon'\, p(\varepsilon') \tag{5.5}$$

We can determine the function $\rho(\varepsilon)$ as follows. If an extra density $\Delta\rho$ of particles is added, this causes $\varepsilon(\rho)$ to shift by an amount $\Delta\varepsilon$, which is such that

$$\Delta\rho = -\Delta\varepsilon\, p(\varepsilon)\, f^{Q}(\varepsilon) \tag{5.6}$$

[$\Delta\varepsilon$ is negative, since the energy $-\varepsilon(\rho)$ increases]. In the limit $\Delta\rho \to 0$ the expression (5.6) leads to a differential equation for $\varepsilon(\rho)$ from which with the aid of (5.4) the unknown function $f^{Q}(\varepsilon)$ can be eliminated. It then becomes

$$\frac{d\log P(\varepsilon)}{f\rho} = -\frac{1}{\phi(\rho)} \tag{5.7}$$

Upon inserting Widom's explicit result (5.3) for $\phi(\rho)$ in (5.7), we can integrate and obtain the desired relation between ε and ρ as

$$P(\varepsilon) = 1 + \tfrac{1}{2}\log(1-2\rho) \tag{5.8}$$

We can now combine Eqs. (5.2)–(5.5) and (5.8) to express the quenched-state energy per site as

$$e^{Q} = -\int_0^1 dP(\varepsilon)\, \varepsilon e^{2[P(\varepsilon)-1]} \tag{5.9}$$

Here the function $P(\varepsilon)$ is still general. If we make use of the explicit expression (1.2) for the distribution of the ε's and transform to the integration variable $u = e^{-\varepsilon}$, we find

$$e^{Q} = \int_0^1 du \log u\, e^{-2u} \tag{5.10}$$

which is Eq. (1.9b).

Finally, from (5.3)–(5.5), (5.8), and (1.2) we obtain for $f^{Q}(\varepsilon)$ the explicit result

$$f^{Q}(\varepsilon) = e^{-2e^{-\varepsilon}}, \qquad \varepsilon \geqslant 0 \tag{5.11}$$

which, in analogy to (3.18), satisfies

$$\int_0^\infty d\varepsilon\, e^{-\varepsilon} f^{Q}(\varepsilon) = \rho^{Q} \tag{5.12}$$

Comparison of the large-ε expansions of the expressions (3.17) and (5.11) shows that, as had to be expected, in the quenched state a larger fraction of sites with very low energies is filled than in the ground state. It is perhaps surprising that also the sites with energies *near zero* have a larger probability to be occupied in the quenched than in the ground state. Since the ground state has a slightly higher density than the quenched state, it follows that the graphs of $f^{G}(\varepsilon)$ and $f^{Q}(\varepsilon)$ must intersect in (at least) two points, $f^{G}(\varepsilon)$ being larger for an intermediate range of energies.

6. CONCLUSION

We have investigated the zero-temperature properties of a one-dimensional lattice gas of particles interacting via a nearest neighbor exclusion potential and placed in a randomly site-dependent external field. The model is a special limiting case of the random field Ising chain that was studied, in particular, by Nieuwenhuizen and Luck.[1] With the aid of the integral equation technique we have determined the ground-state energy and density, as well as the energy-density correlation on a state. Furthermore, we have calculated the pair correlation function; the structure factor

was shown to be a squared-Lorentzian, characteristic of random field magnets. Our solution makes it possible to calculate analytically *all higher order correlations* by the evaluation of a multiple integral. Finally, using work by Widom,[4] we have compared some of the properties of the ground state with those of the quenched state that one obtains by sequentially occupying the lattice with particles, filling each time the site with lowest potential available.

Although this work shows that many of the ground-state and quenched-state properties of this system are well understood, we wish to point out that there still remain quantities of interest that are not easily calculated within the framework given. Among these are in particular quantities that compare the ground state and the quenched state, such as the fraction of sites having the same occupation number in both. Our knowledge of this simple model problem is therefore not yet complete.

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